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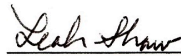
Continuous Opinion Dynamics on an Adaptive Network

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Mathematics from
The College of William and Mary

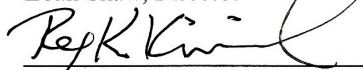
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Continuous Opinion Dynamics on an Adaptive Network

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Abstract

We present a model for opinion dynamics on a network. Opinions are assumed to be a continuous variable, reflecting a possible spectrum of real world opinions. A node's opinion is updated to become more similar to a randomly selected neighbor's opinion, provided that the neighbor's opinion differs by less than a threshold. Initially considering a static network, we establish criteria to determine whether consensus or clustering will be the outcome of the dynamics and on what time scales these states will be reached. We find that smaller step size of opinion update will facilitate consensus formation in the network. Next, in contrast to the static networks with fixed structures, we incorporate the changing nature of the interpersonal relations in real-life social networks. In addition to the opinion dynamics, links that do not communicate due to divergent opinions may be broken with some probability and new links are randomly created. In this way, the network changes in an adaptive manner, which combines the topological evolution of the network with dynamics in the network nodes. Our investigation reveals that adaptation fosters the formation of larger clusters at small mean degrees, while it promotes the division of the major cluster at comparatively large mean degrees.

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Chapter 1

Introduction

People exchange opinions through their social connections constantly. If we think of everyone as a *node*, or connection point, on a social network, each individual can influence his friends through the social connections, and gets influenced by other nodes around him at the same time. This process of opinion exchange accounts for the basics of opinion evolution, and is further complicated by the changing nature of interpersonal relations. For example, an existing connection, or *link*, between individuals can be broken when they hold too divergent opinions, and new connections might be formed due to similar positions. Therefore on the one hand, an individual's opinion is influenced by those in his neighborhood, while on the other hand, he may change his link connections and influence the structure of a network [1]. In this way, the node opinions evolve together with the network structure, and we refer to this system of mutual interaction as dynamics on an *adaptive network*. An analogy for this coevolutionary relation is the national highway system [2]. If we take the pattern of roads as the topology of the network and towns as the nodes linked by them, we know that the roads influence the dynamics of the towns. But if there is constant traffic congestion in certain towns, new roads are likely to be built to solve the problem. In other words, adaptive networks combine the topological evolution of a network with dynamics in the network nodes, and we are interested in how adaptation contributes to opinion formation in social networks.

1.1 Definitions and Notation

First we define some terms in network theory. A *network* G is composed of N nodes and M connecting links. We call a network *undirected* if nodes are connected by links symmetrically, that is, there is no direction associated with each link. The *degree* k of a node refers to the number of nodes connecting to it and we use *mean degree* $\langle k \rangle = \frac{2M}{N}$ to measure the overall connectivity of a network [3]. Applying these definitions to a real-life social network, we think of a system where each node corresponds to an individual, exchanging his opinion through the links of social connections. If we consider friendship as a mutual relation, then the social network is on an undirected graph, in the sense that an opinion can spread along links in both directions. The number of friends that an individual has corresponds to his degree in the network, and the mean degree of the system shows how many friends each individual has on average, thus implying how connected the social network

is on average.

We look at two types of network structures in this thesis: complete graphs and Erdos-Renyi graphs. A *complete graph* is a simple undirected graph in which every pair of distinct nodes is connected by a unique link. By definition, for each node in a complete graph its degree k is $N - 1$. An *Erdos-Renyi graph* refers to a random network generated with specified properties. In this thesis we specify the desired number of nodes and links in a network, and generate an Erdos-Renyi graph by choosing uniformly at random from the collection of all graphs that have the specified number of nodes and links. For example, given node number $N = 3$ and link number $M = 2$, we uniformly pick one out of the three possible graphs with 3 nodes and 2 links, and each of them will be included with a probability $\frac{1}{3}$.

1.2 Opinion Modeling on Networks

Much previous research looks at the process of opinion formation on social networks. In the early studies, opinions are represented as a discrete variable on networks with fixed structures. For instance, in the model proposed by Axelrod [4], individuals with a set of cultural features are placed at fixed sites of a social network. The model updates opinions by following the rule that the more similar a node is to a neighbor, the more likely that the node will adopt one of the neighbor's traits.

This type of discrete static models becomes more realistic with the incorporation of changing network structures. In a study conducted by Holme and Newman [5], they consider a potentially very large sparse network changing adaptively with nodes holding discrete opinions. At every time step, they let the system either randomly pick an existing link and move it to lie between two individuals whose opinions agree, or randomly choose a node and change its opinion to agree with that of one of its neighbors. These two types of updates correspond to the topological evolution of the network and the dynamics of opinion exchange in an adaptive network, respectively. They introduce a rewiring probability parameter ϕ to control the relative speed of the two processes and examine the phase transition of the dynamics as the parameter ϕ varies. Based on Holme and Newman's model, Malik and Mucha make more realistic modifications and propose to incorporate the effect of heterogeneous influences and tendencies to local clustering in the dynamics [6]. Instead of a constant rewiring probability ϕ , they assign ϕ values according to a probability distribution for individuals to accept opinions, and replace the random rewiring step with a step that prefers rewiring to nodes that are already close within the network. In this way, their model accounts for the heterogeneity of influence for individuals in social networks and the phenomenon that friends of friends have a higher likelihood of also being friends. Another important category under discrete adaptive models is the well-studied coevolving binary voter model, in which each node as a voter has a choice between two political parties (0 or 1). These studies examine the properties of opinion evolution with various definitions of opinion exchange and link rewiring rules to account for real-life social networks under different circumstances [7][8].

While discrete opinion models can be accurate in modeling opinion formation in certain scenar-

ios, another refinement considers the diversity of real-world opinions and represents opinions as a continuous variable on networks instead. This extension from discrete opinions to the continuous case makes the studies of opinion dynamics more realistic, because in real world opinions can be too diverse to model as a discrete variable. For example, opinions can be how much we believe in what our friend says, or how politically liberal an individual is. In these scenarios, opinions actually take values on a spectrum, and thus continuous opinion variable might be a better fit. Weisbuch’s model, with opinions uniformly distributed on the interval of $[0, 1]$, is an early work of continuous opinions on static networks [9]. In his model, a node communicates with its neighbor only if their opinions are close enough. The rationale for this notion of *bounded confidence* is that individuals prefer to interact when their opinions are already relatively similar; otherwise, they will not be interested in socializing. At each time step, a node is randomly chosen to communicate with one of its neighbors, and whenever their opinion difference is below a given threshold, the node’s opinion x is updated as $x_{new} = x + \alpha \cdot (y - x)$, where y is the opinion of the neighboring node and α is the convergence parameter. The opinion of the neighboring node y is updated similarly as $y_{new} = y + \alpha \cdot (x - y)$. Mostly simulating on static complete networks, Weisbuch concludes that high thresholds yield consensus in the population, whereas low thresholds result in several opinion clusters.

Little existing research has looked at the dynamics of a continuous opinion variable on adaptive networks. However, combining diversified opinions with the coevolution of node opinions and network structures, continuous opinion dynamics on adaptive networks can be of great importance in modeling some processes of opinion formation realistically. In a continuous adaptive model proposed by Kozma and Barrat [10], they focus on studying the effect of different values of communication threshold on the networks, and how adaptation influences the role of bounded confidence on the final state. They control the relative frequencies of the two processes of opinion exchange and link rewiring by a parameter $\omega \in [0, 1]$. At each time step, a node i and one of its neighbors j are chosen at random. With probability ω , an attempt to break the connection between i and j is made: if their opinion difference is above the threshold of bounded confidence, a new node k is chosen at random and the link (i, j) is rewired to (i, k) . With probability $1 - \omega$ on the other hand, the opinions evolve according to the same rule in Weisbuch’s model if they are within the bound of confidence. By comparing the behaviors of static networks and adaptive networks, they conclude that the adaptation of the network topology fosters cluster formation by enhancing communication between nodes of similar opinion. However, large clusters can be more easily broken by rewiring, and global consensus is more difficult to reach.

1.3 Overview

Similar to Kozma’s model, our work looks at the evolution process of continuous opinions on adaptive networks, and how adaptation influences the final state of convergence. In this thesis, we use Weisbuch’s model [9] as a starting point and extend their original model by adjusting the network structure from complete graphs to Erdos-Renyi graphs. A key theme throughout this work is to explore what features influence the formation of consensus in the population, and results

about related features on static networks are discussed in Chapter 2. Next in Chapter 3, in contrast to the static case, we implement link rewiring and allow the dynamics to evolve adaptively. We compare the behavior of adaptive networks to that of static networks, and investigate the impact of adaptation on the dynamics. Because most previous research study the effect of communication threshold on the opinion formation process [9][10], little is known about the role that convergence parameter α plays. We examine the effect of α on both static networks and adaptive networks, and demonstrate how convergence parameter α , as a measure of how quickly individuals change their opinions influences the final state of convergence. We also explore the impact of network geometry by changing the mean degree of Erdos-Renyi networks. Finally in Chapter 4, we conclude the results of the study and propose potential directions to work on in the future.

Chapter 2

Static Network

In this chapter, we examine static networks with various graph structures. Under this setting, each node can synchronize its opinion through the process of communication, whereas its links are not allowed to be changed. Thus nodes exchange opinions, while network structures are fixed.

2.1 Node Opinion Update

Consider a network of N nodes. Here we implement the same rule for updating node opinion as in Weisbuch's model [9]. We let initial node opinions be uniformly distributed on $[0, 1]$. For each update, we pick a node at random and then pick one of its neighbors to listen to at random. The two nodes exchange their opinions when their difference in opinion is smaller in magnitude than a communication threshold d . That is, suppose the two nodes have opinions x_i and x_j .

If $|x_i - x_j| < d$,

$$\begin{aligned}x_{i,new} &= x_i + \alpha \cdot (x_j - x_i) \\x_{j,new} &= x_j + \alpha \cdot (x_i - x_j)\end{aligned}\tag{2.1}$$

where α is the convergence parameter.

Note that the convergence parameter $\alpha \in [0, 0.5]$, where $\alpha = 0$ means that individuals refuse to synchronize their opinions, while $\alpha = 0.5$ allows two nodes to synchronize their opinions to be the same within one update. We let every time step consist of N updates.

Next, we define convergence as follows. For all $i - j$ neighboring pairs, the network is converged if $|x_i - x_j| \geq d$ or $|x_i - x_j| < 10^{-4}$. Here 10^{-4} is an arbitrary small positive real number. With a different arbitrary value, the network might have a different convergence time, but its qualitative behavior and final state after convergence would be approximately the same. By defining convergence in this way, the communication process will terminate when each neighboring pair either holds too different opinions to communicate or has already been synchronized to be similar enough. The introduction of convergence here is to examine the convergence times of network evolution, which will be discussed more in later sections.

2.2 Complete Network

We start with complete networks, in which each node is connected to every other node in the graph. Following the update rule in Section 2.1, we simulate the evolution of opinions with different communication thresholds d (shown in Figure 2.1). The upper figure with communication threshold $d=0.5$ reaches a final state of consensus, while the lower one with $d=0.2$ evolves into two clusters at the end. The result of simulations implies that when individuals are more willing to talk to others with different opinions, the population is more likely to reach a final state of consensus. In other words, the extent of open mindedness of a society contributes directly to its opinion formation and its chance of reaching consensus or polarized opinions.

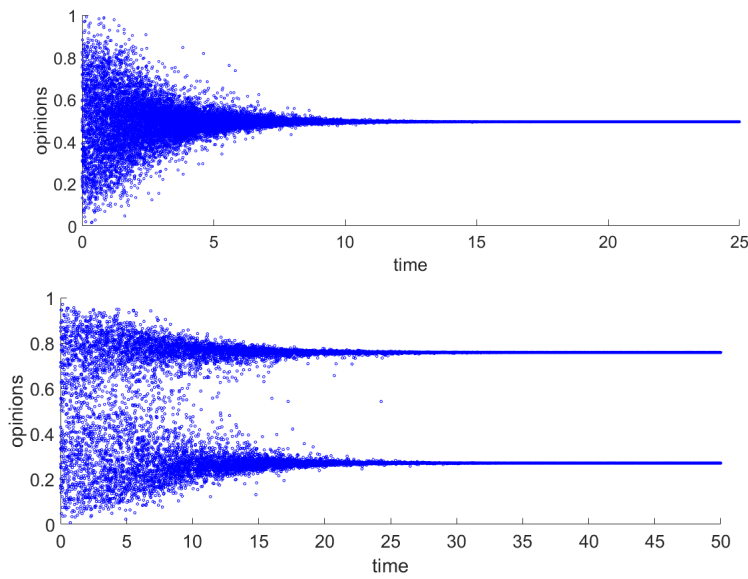


Figure 2.1: Opinion evolution of complete graphs (Upper: $d=0.5$, $\alpha = 0.5$, $N=2000$; Lower: $d=0.2$, $\alpha = 0.5$, $N=1000$)

Similar simulations were carried out before by Weisbuch [\[9\]](#). It should be noted that both Weisbuch’s work and Figure 2.1 plot only the two nodes that are selected and updated at each iteration. With a different plotting algorithm of showing all the nodes at every time step, the opinion evolution figure may look very different, as shown in Figure 2.2. Although here we depict every node in the network at each update, everyone except the two that are selected stays unchanged. This explains the characteristic horizontal bars that we find in Figure 2.2. A bar continues as long as the node has not yet been selected and updated. Chances are that a few nodes are either not picked or not similar enough to their neighbors to communicate. These nodes hold their opinions unchanged throughout the iterations and thus are not represented in Figure 2.1. Furthermore, Figure 2.2 demonstrates that the final state of two clusters of opinions does not guarantee that all the nodes are included in one of the two clusters. Although the majority converges, there can be some leftover nodes floating outside the big two. In fact, this explains why we see four “clusters” in Figure 2.2: two of them are not major clusters, but some bits of nodes that are forever left out. They will not be synchronized into one of the major clusters even with more time, because their opinions are already too far from the two major clusters to communicate. We will refer to these

nodes as *floating nodes* or *floating clusters*, in contrast to major clusters shown in Figure 2.1. The number of floating clusters in a network is random, depending on the initial network configuration and the sequence of communication. In complete networks, we have not observed clusters of intermediate size at the final state of convergence. The difference between floating clusters and major clusters appears clear-cut. Empirically, the number of floating nodes is below 10 in a complete graph of size 1000. The result of major clusters versus floating clusters is further demonstrated in Figure 2.3, where we plot the opinion distribution of one simulation. The tiny bars at opinion 0 to 0.05 and opinion 0.5 to 0.55 suggest the existence of floating nodes outside the two major clusters.

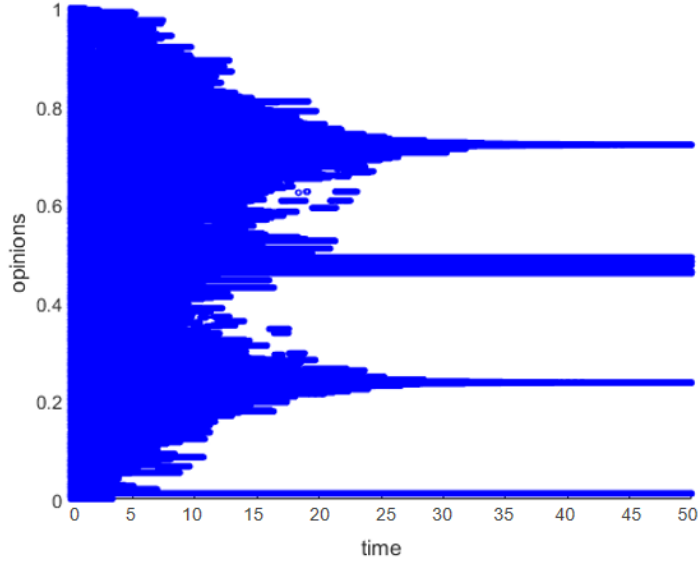


Figure 2.2: Opinion evolution of complete graphs with all the nodes plotted ($d=0.2$, $\alpha=0.5$, $N=1000$)

Next, we examine the convergence time of complete graphs with varying network sizes. With fixed convergence parameter α and communication threshold d , Figure 2.4 demonstrates that in general the convergence time increases linearly as the network size gets larger, and the best fitted line intercepts near the origin. Recall that we let every time step consist of N updates, where N is the number of nodes in a network. Intuitively, we might expect a fixed convergence time for complete graphs of different sizes, because the increase in convergence time for larger graphs can be “compensated” by the way of counting time. However, what should be noted is our method of counting time does not account for the impact of different mean degrees. During the process of communication, for every pair of nodes that needs to be synchronized in the network, we get to choose one node every $\frac{N}{2}$ updates, that is, every $\frac{1}{2}$ time step. Then we can pick its correct neighbor to listen to with probability $\frac{1}{\langle k \rangle}$. These two steps imply that the typical communication time for a given pair of nodes in the network is $\frac{\langle k \rangle}{2}$. It suggests that the convergence time of complete graphs is proportional to the mean degree of the network, which is approximately N , and thus explains why we see the increase in convergence time as the complete graph gets larger.

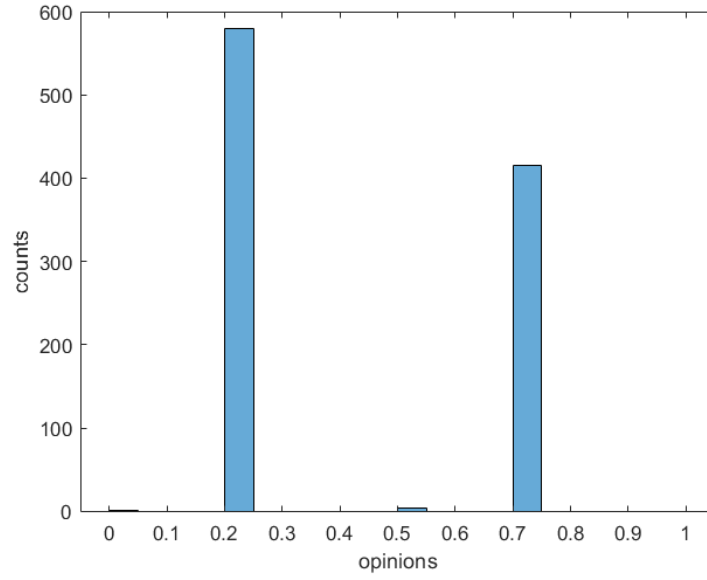


Figure 2.3: Histogram of opinion distribution ($d=0.2$, $\alpha=0.5$, $N=1000$)

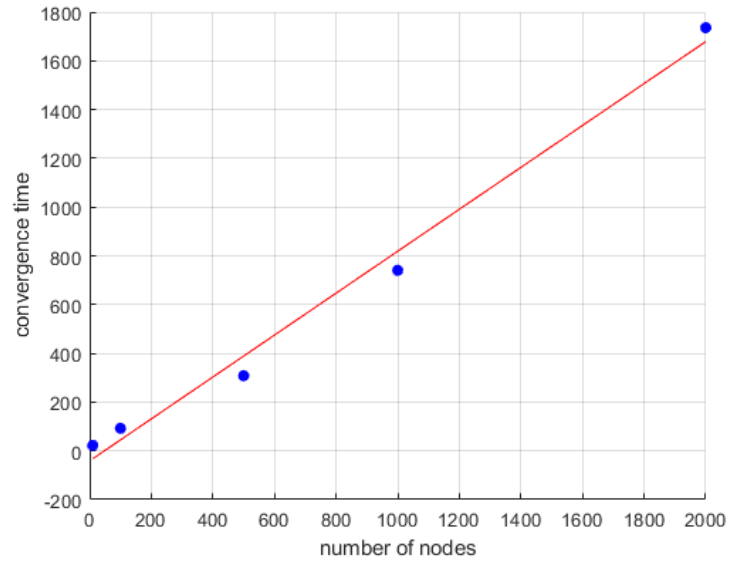


Figure 2.4: Convergence time of complete graphs with varying network size, averaged over 20 replications ($d=0.2$, $\alpha=0.5$)

2.3 Erdos-Renyi Network

So far we have been working on complete graphs. However, it is not likely that everyone knows each other in a society. Real-life social networks are, in fact, far more sparse than complete graphs [11]. Therefore in this section, we change the network structure to Erdos-Renyi graphs, in which we specify the number of nodes and edges of the graph but the specific network configuration is randomly determined.

To understand the effect of graph structure on opinion evolution, we explore the dynamics from two perspectives: the convergence time of the network and the final state of the network after it is fully converged. We follow the node opinion update rules specified in Section 2.1.

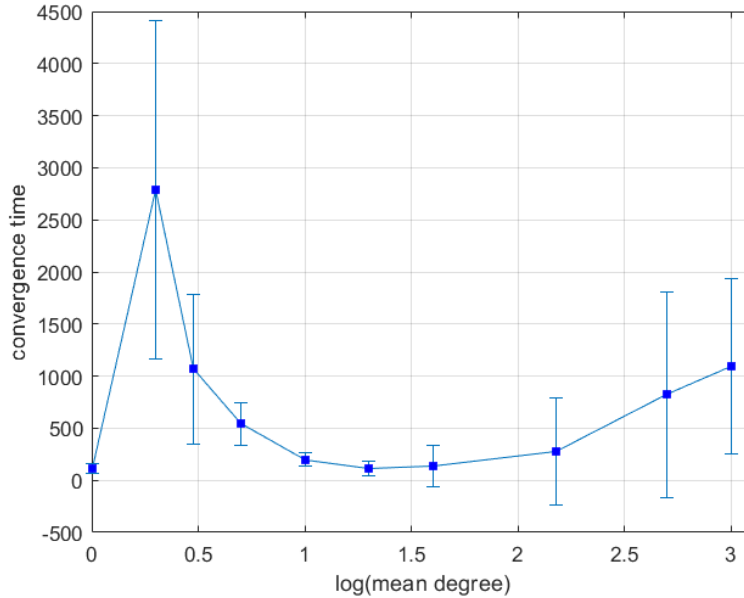


Figure 2.5: Convergence time of Erdos-Renyi graphs with different mean degrees, averaged over 20 replications ($d=0.2$, $\alpha=0.5$, $N=1000$)

Figure 2.5 demonstrates the convergence time of Erdos-Renyi graphs with different mean degrees, and the vertical error bar at each mean degree indicates the standard deviation of convergence time for 20 replications. The figure can be divided into three phases: going from left to right, the convergence time increases from mean degree $\langle k \rangle = 1$ to $\langle k \rangle = 2$, decreases from $\langle k \rangle = 3$ to $\langle k \rangle = 20$, and increases again until the graph is complete. In the first phase, the mean degree of $\langle k \rangle = 1$ can be considered as a trivial case: when the network is so sparsely connected, there is not much communication happening in the system so it doesn't take long for every node to synchronize to those it can communicate with. At $\langle k \rangle = 2$ the network, though still sparse, becomes better connected than the graph at $\langle k \rangle = 1$. On the one hand, this increase in connectivity gives the network more synchronization to process, while on the other hand, its general lack of connectivity, especially when compared with the networks of higher mean degrees, gives the dynamics low communication efficiency. These two factors together increase the convergence time and cause the

steep jump that we see in the first phase. Next, the second phase sees the network continue to become better connected. As the mean degree increases, there exist more paths for every node to communicate with its neighbors, which increases the efficiency of synchronization and thus lowers the convergence time. Finally, in the last phase of mean degree $\langle k \rangle = 20$ and beyond, the network becomes dense enough that its performance resembles that of the complete graph. Thus an explanation similar to the complete networks in Figure 2.4 can be applied here. As in the previous section, we think of the network as pairs of nodes that needs to be synchronized. On average we get to choose one node every $\frac{N}{2}$ updates, or every $\frac{1}{2}$ time step. Because the correct neighbor is picked with probability $\frac{1}{\langle k \rangle}$, the communication time of the network is $\frac{\langle k \rangle}{2}$. Therefore it explains why we see the increase in convergence time as the mean degree of the graph increases.

In Figure 2.5 the standard deviation of convergence time is comparatively large at two places: mean degree $\langle k \rangle = 2$ to $\langle k \rangle = 3$, and $\langle k \rangle = 150$ and beyond. For mean degrees $\langle k \rangle = 2$ and $\langle k \rangle = 3$, we hypothesize that the large standard deviation comes from the big variation of initial network configuration at low mean degrees. Because assigning a small number of links over a fixed number of nodes allows much freedom in terms of network structure, these widely different network structures can eventually lead to the variation in convergence time. While the large standard deviation at low mean degree can be caused by the randomness of graph structures, the error bars at mean degree $\langle k \rangle = 150$ and beyond are related to the floating nodes that we mentioned in the previous section. In simulations we observed that when there is one floating cluster outside the two major clusters, the more nodes the floating cluster includes, the faster the network can get fully converged. Empirically, nodes converge to the two major clusters faster than the floating cluster. This leads to the situation that the major clusters and the floating cluster are already too different to communicate; nodes within the major clusters are fully synchronized but those within the floating cluster are not. Note that in order to terminate the communication process, opinions within the floating cluster also need to be synchronized. When the size of the floating cluster is larger, every node in the cluster has a larger number of “correct” neighbors, nodes in the floating cluster that they can actually communicate with. In other words, when a floating node is selected, the more nodes that exist in the floating cluster, the higher the chance that a randomly selected neighbor can communicate with the first node. With more effective selections, the floating cluster converges faster, and so does the entire network. Thus we conclude that the randomness of initial network configuration and selection sequence together contribute to the uncertainty of the number of floating nodes, which directly determines the convergence time of the network.

Next, we examine the final state of the network after it is fully converged. We looked at 20 replications and then picked one representative that visually looked typical (shown in Figure 2.6). Each circle in the figure represents a cluster in the network. The circle is centered at the opinion value of the cluster and its size is dependent on the log size of the cluster. Going from right to left, the network changes from complete to sparse. The two major clusters gradually lose their nodes due to the decreasing connectivity of the graph, until the network contains only floating clusters at mean degree $\langle k \rangle = 1$. Figure 2.7 also demonstrates this trend by showing the percentage of nodes in the two biggest clusters after the network is synchronized.

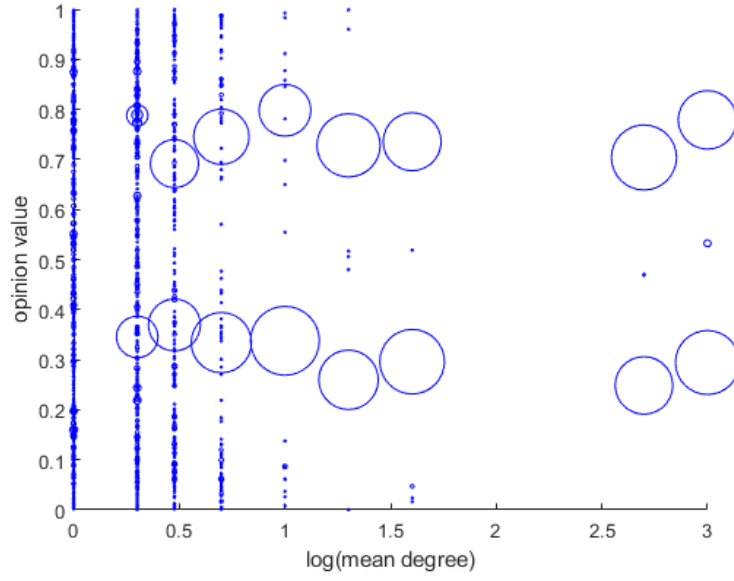


Figure 2.6: Cluster number and size for Erdos-Renyi graphs with different mean degrees ($d=0.2$, $\alpha=0.5$, $N=1000$)

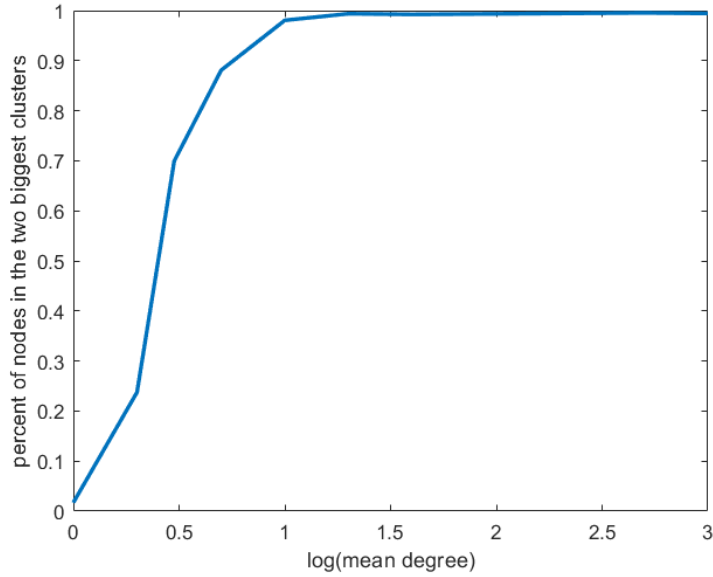


Figure 2.7: Percentage of nodes in the two biggest synchronized clusters for Erdos-Renyi graphs with different mean degrees, averaged over 20 replications ($d=0.2$, $\alpha=0.5$, $N=1000$)

2.4 Convergence Parameter α

Previous studies have examined the effect of communication threshold d on the number of clusters at the final state. In Weisbuch’s paper [9], they proposed a “ $\frac{1}{2d}$ rule”, that the number of clusters at the final state equals the integer part of $\frac{1}{2d}$. In this section, we will instead focus on the convergence parameter α by exploring its effect on the number of clusters at the final state and its relation to the overall opinion difference in the network.

First, we observe that with a fixed value of the communication threshold $d = 0.2$, Edos-Renyi networks of mean degree $\langle k \rangle = 10$ can either reach a final state of consensus or two major clusters of opinions, as the value of convergence parameter α varies. We find that as the value of α decreases, the probability of a network reaching consensus at the final state increases drastically (shown in Figure 2.8). One way to explain this observation is that because a large convergence parameter α

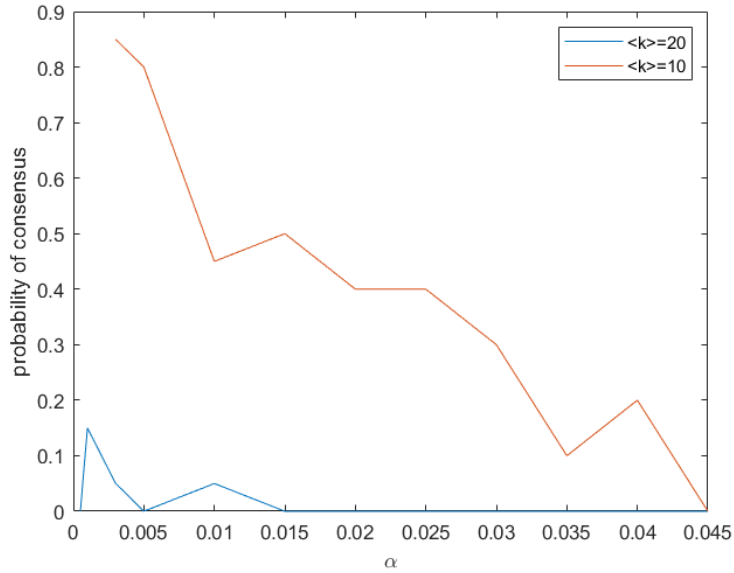


Figure 2.8: Probability of reaching consensus for Erdos-Renyi networks with varying values of convergence parameter α , averaged over 20 replications ($N = 1000, d = 0.2$)

can change node opinions sharply, two nodes that can communicate before might be too different to synchronize after one of them experiences a single update. Small step size of opinion update, on the other hand, avoids the dramatic changes. Two neighboring nodes can still communicate through their connection after the node update, and thus macroscopically the network has a higher probability of reaching consensus at the end. This process of opinion formation is further illustrated by the diagram in Figure 2.9. Imagine a small complete network of 4 nodes with opinions: 0.3, 0.4, 0.5, and 0.6. We let communication threshold $d = 0.2$. With the same initial configuration and the same node updates, the network with $\alpha = 0.5$ reaches a final state of two clusters within two steps, whereas the case with $\alpha = 0.01$ can still potentially reach consensus by following some specific sequences of updates in the next steps (for example, updating the highlighted link in the next step, and so forth). Of course this does not guarantee a final state of consensus at the end, since the choice of which link to update is entirely random. However, compared with large α values,

small values of convergence parameter α makes consensus more likely to happen.

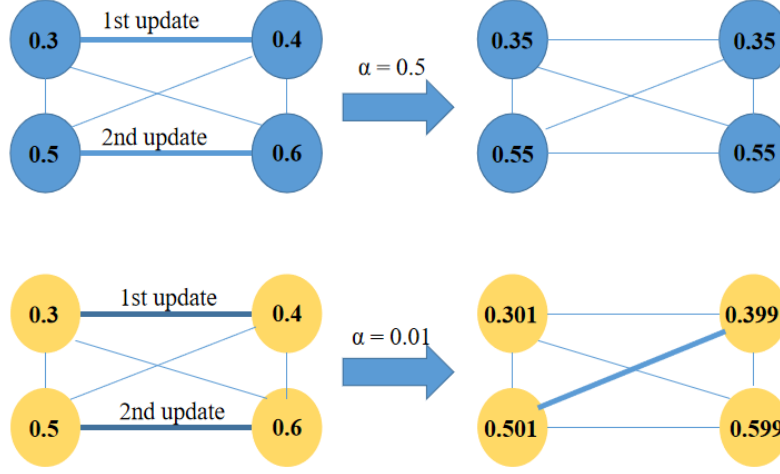


Figure 2.9: Diagram of opinion evolution process for convergence parameter $\alpha=0.5$ and 0.01 , communication threshold $d = 0.2$

It should be noted that although we expect to see roughly the same trend for all Erdos-Renyi networks, the transition from two major clusters to consensus that we observed in networks of mean degree $\langle k \rangle = 20$ doesn't follow the same tendency (shown in Figure 2.8). The highest probability of reaching consensus is at $\alpha = 0.001$, and we only find 3 final states of consensus out of 20 replications. It could be the case that the transition happens at even smaller values of convergence parameter $\alpha < 5 \times 10^{-4}$ that we didn't examine due to computational limitations. Or possibly the number of clusters at the final state relates to the mean degree of Erdos-Renyi networks. We need further investigation to figure out the underlying mechanism, but overall we demonstrate that the convergence parameter α does play a role in determining the final state of a network, and the " $\frac{1}{2d}$ rule" proposed by Weisbuch holds only if the value of convergence parameter α is fixed at 0.5.

Next, we examine the relation between convergence parameter α and the overall opinion difference in complete networks. Here we use a parameter $\sigma = \sqrt{\frac{1}{2N} \sum_{i=1}^N \sum_{j=1}^N (x_i - x_j)^2}$ to measure the overall difference of opinions in the network and we only look at the case of communication threshold $d = 0.5$ to avoid the influence of two clusters at the final state. In other words, we know that as time goes to infinity, nodes in the network will converge to consensus for any value of convergence parameter α , and the parameter σ approaches 0. We are interested in how the overall opinion difference of the network σ , as time progresses, relates to the value of α through the process of communication.

Following the node update rule specified in Section 2.1, a node opinion differs from itself in the previous time step by its opinion difference with its neighbor in the previous time step times the convergence parameter α , that is,

$$x_{i,t+1} - x_{i,t} = \alpha(x_{j,t} - x_{i,t}). \quad (2.2)$$

Assume opinions change little at every update. Equation 2.2 can be written as

$$\frac{d}{dt}x_i \approx \alpha(x_j - x_i). \quad (2.3)$$

This implies that individual opinions and the overall opinion difference of the network σ are roughly subject to exponential decay with rate α . If so, we expect to see $\log(\sigma)$ decreasing approximately linearly in time with a slope of $-\alpha$.

Figure 2.10a demonstrates the time-series result of overall opinion difference σ with varying values of α . Although each curve in general decreases linearly in Figure 2.10a, it should be noted that each line of opinion evolution is composed of two parts: a short-time flatter segment followed by another steeper one as time increases. This segmentation can be seen more clearly in curves of opinion evolution with smaller α . We don't understand why the exponential decay has two different rates, or exponents. It remains a mystery that needs further investigation.

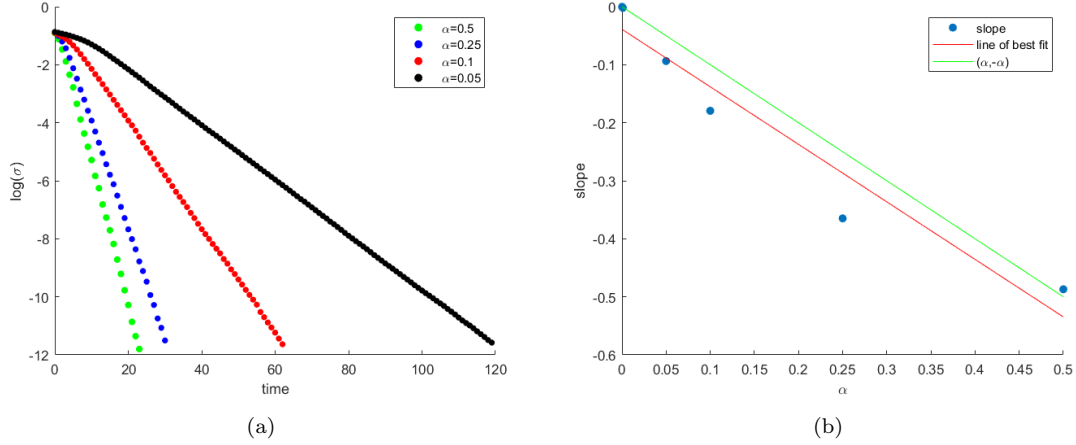


Figure 2.10: Relation between overall opinion difference σ of complete networks and convergence parameter α ($N=1000$, $d=0.5$). Panel (a) demonstrates the time-series result of overall opinion difference $\log(\sigma)$ with varying values of α . Panel (b) finds the slope of the second segment for each curve of opinion evolution in panel (a), and compares the best fit line of the slopes with the line of $(\alpha, -\alpha)$.

Next we find the slope of the second segment for each line of opinion evolution, and draw the line of best fit on the scatter plot of slopes (shown in Figure 2.10b). Visually comparing the line of best fit to the line of slope $-\alpha$, we conclude that the best fit line is of slope approximately $-\alpha$, and thus the overall opinion difference of the network σ roughly follows exponential decay as we hypothesized.

It should be noted that the line of best fit compares the “averaged” slope of the curves of opinion evolution, and finds it close to $-\alpha$. When we compare $\log(\sigma)$ with each α separately, the slope, or exponent of the exponential decay is not necessarily consistent with our rough expectation. Figure 2.11 demonstrates the discrepancies that we can see in comparison, and we don't know the reason for the exact observed discrepancies.

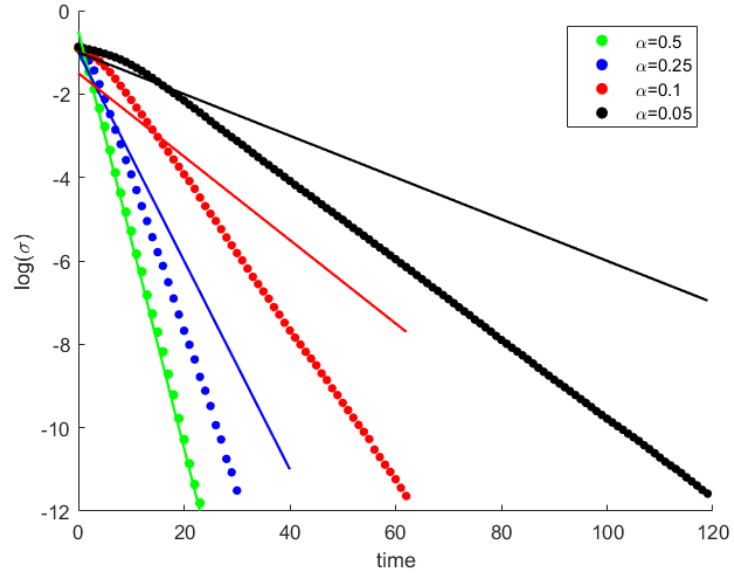


Figure 2.11: Points of overall opinion difference σ from Figure 2.10a, with lines of slope $-\alpha$ overlaid for comparison. The intercepts are arbitrary.

Chapter 3

Adaptive Network

In the previous chapter, we explored the behavior of static networks, in which each node synchronizes its opinion on a fixed graph. In this chapter, we incorporate the changing nature of interpersonal relations and look into a system of an adaptive network, where each node is allowed to change both its opinion and links to other nodes in the network. Thus through the process of communication, the network changes in an adaptive manner, which combines the dynamics in the network node opinions with the topological evolution of the network.

3.1 Link Update

Imagine a real-life social network. If we think of friends as a pair of neighbors on a graph that can potentially communicate, every individual has close friends that he actively communicates with, and other distant acquaintances with whom he barely exchanges opinions. Instead of maintaining these distant relations, we are more likely to quit ineffective social connections and seek new friendship somewhere else. Thus the structure of real-life social network changes together with the process of opinion evolution [1]. To model the scenario described above, we introduce link update and define the update rule as follows. Pick a link at random from the list of links. When the nodes on each side of the link don't communicate, that is, their opinion difference is not below the communication threshold d , delete the link with probability p . If the non-communicating link is successfully deleted, create a new link anywhere in the network at random.

Note that we avoid duplicating links that already exist or creating links from a node to itself. So the link update rule guarantees that the total number of links in the network stays constant. A new link is created randomly only if a non-communicating link is deleted. We follow the same node opinion update rule specified in Section 2.1, and alternate between node update and link update at each time of update. As in the static network, every time step consists of N updates, which are now N node updates and N link updates, where N is the number of nodes in the network.

Because now nodes can get rid of undesirable connections, the convergence defined in Section 2.1, which allows the existence of non-communicating links at the final state, is no longer applica-

ble. So we redefine convergence for adaptive network as follows. For all $i - j$ neighboring pairs, if $|x_i - x_j| < 10^{-4}$, then the network is converged. As in the static case, 10^{-4} is an arbitrary small positive real number. With a different arbitrary cut-off value, the qualitative behavior of the system would still be similar. It should be noted that this new definition of convergence is stricter than the one in Section 2.1. Rather than allowing each neighboring pair to be either synchronized or too different to communicate at final state, the new convergence definition requires that all existing links are effective and the opinions of connected nodes are fully synchronized. Thus going from static networks to adaptive networks, our definition of cluster changes slightly. In the former case, when we conclude that the network converges to two major clusters of opinions, the two clusters might be actually connected together in terms of physical structure. When we call them “separate”, we are emphasizing their opinion difference. A cluster in adaptive networks, however, is guaranteed to be structurally separated from the outsiders by our definition of convergence.

3.2 Two Extreme Cases

To examine the effect of incorporating link update, we start with two extreme cases: the case of maximum node update speed and slow link update speed, and the case of maximum link update speed and slow node update speed. Any other combinations of node versus link update speed hypothetically have behavior somewhere between these two extremes. Recall that the step size of opinion synchronization at each update is determined by the convergence parameter $\alpha \in [0, 0.5]$, whereas the link update speed is controlled by the probability p , the chance of deleting a non-communicating link and replacing it with a potentially effective one. Therefore, the two extreme cases discussed in this section will be implemented by the extreme values of α and p . For each of the two cases, we look at its number of link changes through the communication process and the percentage of communicating links in the network. Example simulation results are shown in Figure 3.1.

For the extreme case of maximum node update speed and slow link update speed, we let convergence parameter $\alpha=0.5$ and rewiring probability $p=0.001$. As when we define convergence, the value of $p=0.001$ is an arbitrary small positive number. It implies that on average, the network of size $N = 1000$ can only replace 1 non-communicating link every 1000 link updates, or every 1 time step. This inefficiency of link update is illustrated by Figure 3.1, where we can see that the number of link changes is very small throughout the communication process, and the percentage of communicating links in the network increases slowly, since the network cannot quickly fix those non-communicating links. Not surprisingly, the slow link update speed leads to the domination of node update so that to some extent, we can think of this extreme case as approximately a static network. The final state of the network after it is fully converged does show similarity to that of a static case. Recall that in static networks of mean degree $\langle k \rangle = 10$ and communication threshold $d = 0.2$, we find two major clusters of about the same size at the final state of convergence, one cluster with opinions around 0.3, and the other approximately 0.8 (see Figure 2.6). In this extreme case with rapid node update and slow link update, we observe the same results mentioned above. The biggest difference, however, is that the adaptive network contains a smaller number of floating nodes at the final state compared with its counterpart in the static case. This observation can be

explained as follows. In the communication process of static networks, when a node holds a too different opinion from all of its neighbors, there is no one it can further communicate with to update opinions. Thus it will be forever left out as a floating node. If this node is in an adaptive network, however, its opinion evolution is far from termination. As the network breaks non-communicating links and randomly creates new connections, possibly the left-out node will be reconnected to a node that it can actually communicate with. Serving as a bridge between the floating node and one of the major clusters, the node in the middle can potentially drag its neighbor closer towards the major cluster, and eventually get it incorporated into one of them. Therefore, by enhancing the communication between nodes with similar opinions, adaptation reduces the number of floating nodes in the network, and promotes the size of major clusters. Similar results about the effect of adaptation are also observed in Kozma's model [10].

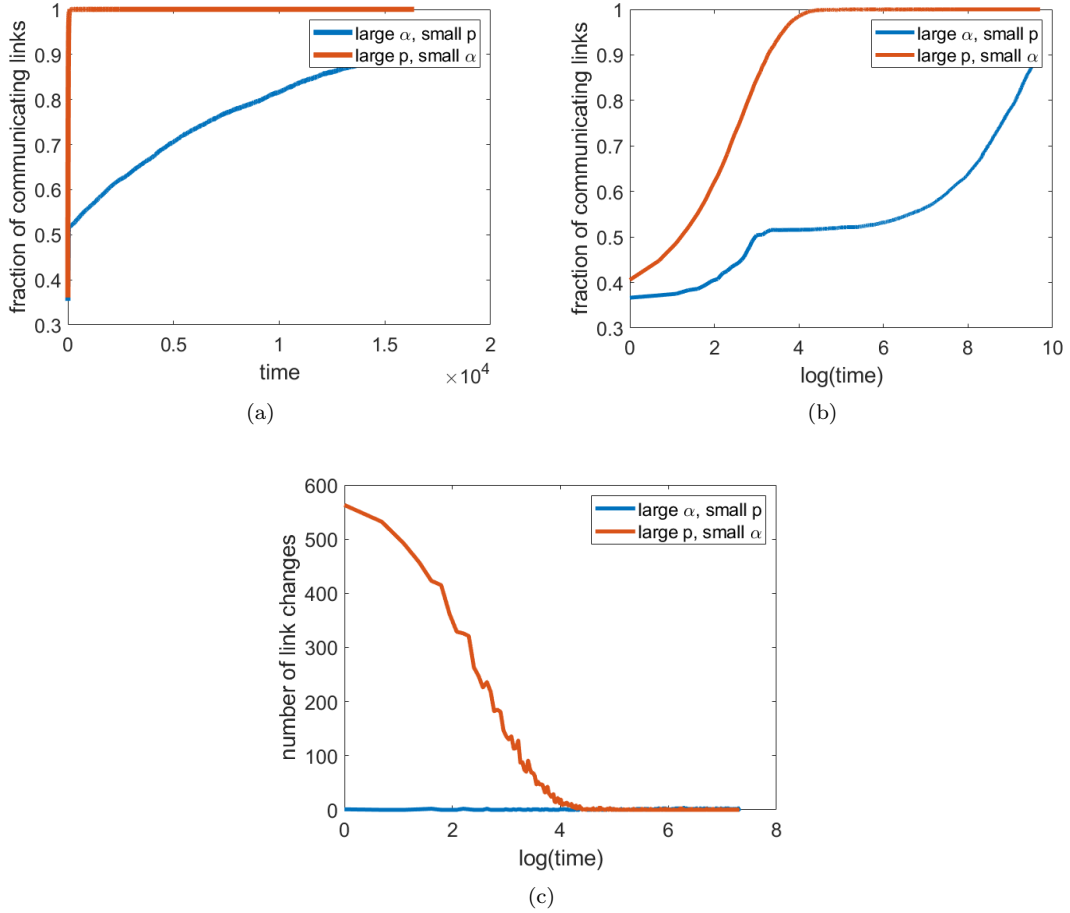


Figure 3.1: Panel (a) demonstrates the fraction of communicating links versus time in both extreme cases. Panel (b) is the same simulation in Panel (a), but with the natural log of time to show the changes at the beginning of the evolution. Panel (c) shows the number of link changes per time step with time on a natural log scale. Note that we truncate the evolution time because the tendency stays the same, so the ending time in this Figure is not the convergence time of the networks. ($d = 0.2$, $N = 1000$, mean degree $\langle k \rangle = 10$)

For the other extreme case of maximum link update speed and slow node update speed, we

let probability $p = 1$ and convergence parameter $\alpha = 0.001$. This setup allows the network to immediately delete a non-communicating link whenever it is selected for update. As shown in Figure 3.1, the number of link changes is very large at the beginning, and then quickly drops down and maintains a low level because the network is capable of fixing most non-communicating links efficiently at the beginning of the evolution. For the same reason the percentage of communicating links in the network increases rapidly to almost 1 within 60 time steps. This network reaches a final state of consensus because the convergence parameter α has a very small value. It relates back to our discussion in Section 2.4 about the effect of convergence parameter α on the number of clusters at final state for static networks. Although here the network changes adaptively, the impact of small convergence parameter still exists.

It should be noted that in both extreme cases, the convergence time for adaptive networks is far longer than that of static networks. In the two example simulations shown in Figure 3.1, the convergence time is $t = 85678$ for the case of maximum α and small p , and $t = 86021$ for maximum p and small α . This observation is consistent with our stricter definition of convergence for adaptive networks, which requires both opinion synchronization and link rewiring.

In the next section, we will revisit the role of convergence parameter α and explore how the incorporation of link update influences the effect of convergence parameter on a network. Because a slow speed of link update will, to some extent, make the opinion evolution similar to that of static networks, in the rest of this chapter we will let the link update probability $p = 1$ to see the maximum effect of adaptation on the behavior of a network.

3.3 Revisit: Convergence Parameter α

Recall that in static networks, the number of clusters at the final state is dependent on the value of communication threshold d , the convergence parameter α , and potentially the mean degree of an Erdos-Renyi network. Therefore to examine the effect of convergence parameter α alone on adaptive networks, we fix the communication threshold $d = 0.2$ and mean degree $\langle k \rangle = 10$, and look at the probability of reaching consensus for adaptive networks with varying values of α (shown in Figure 3.2).

Similar to static networks with the same communication parameters, the adaptive networks here converge either to consensus or a final state of two clusters. However, compared with the behavior of static networks, the final state of adaptive networks transitions from two clusters to consensus at larger values of convergence parameter α . This observation suggests that the introduction of link update allows the network to reach consensus with comparatively large α values that would cause a final state of two clusters without adaptation. In other words, the incorporation of adaptation facilitates achieving consensus in the population. We use the diagram in Figure 3.3 to explain this consensus formation process. Let us imagine an evolving adaptive network that is about to separate into two independent clusters. After deleting a non-communicating link in the network, the link update might randomly place a new connection between two nodes from differ-

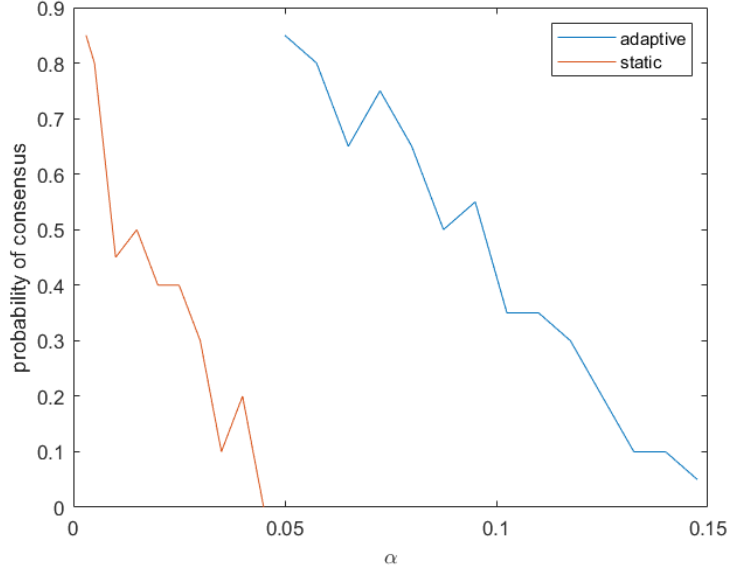


Figure 3.2: Probability of reaching consensus for static and adaptive networks with varying values of convergence parameter α , averaged over 20 replications ($d=0.2$, $N=1000$, mean degree $\langle k \rangle = 10$, $p = 1$ for adaptive network)

ent clusters. Although their opinion difference is close to the communication threshold d , the two nodes can still exchange opinions and thus bring themselves closer towards each other. This step of synchronization can potentially influence the configuration of the entire network deeply because each of the two nodes can cause a chain reaction for updating the opinions in its cluster through the connecting vertices. That is to say, with one newly-created effective link, the general opinions of both clusters can be changed. Macroscopically, these two nodes can potentially serve as a bridge between the two almost-separate clusters, and allow them to be reunited. Therefore, we find that adaptation fosters consensus formation on networks by enhancing communication between nodes with similar opinions.

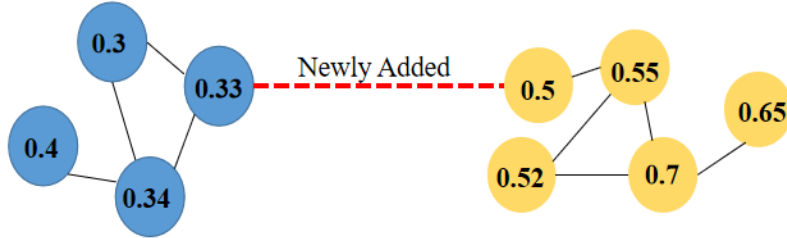


Figure 3.3: Diagram of the consensus formation process in adaptive networks with $d = 0.2$

3.4 Adaptive Convergence Time

Similar to the case of static graphs, we are interested in the convergence time of adaptive networks. In this section, we demonstrate the behavior of adaptive Erdos-Renyi networks with different mean degrees, and draw parallels between Kozma’s paper [10] and our findings to explain the observation.

We focus on sparse networks to examine the effect of link rewiring in adaptive networks, since essentially adaptation makes a sparse network equivalent to a static network with higher mean degree. If the graph is already almost complete, there is not much need to consider adaptation. Figure 3.4 demonstrates the convergence time of adaptive Erdos-Renyi networks with different mean degrees, and the vertical error bar at each mean degree indicates the standard deviation of convergence time for 5 replications. The figure can be divided into three phases: going from left to right, the convergence time increases from mean degree $\langle k \rangle = 1$ to $\langle k \rangle = 3$, decreases from $\langle k \rangle = 3$ to $\langle k \rangle = 5$, and stays approximately the same until mean degree $\langle k \rangle = 40$. Compared with the convergence time of static networks in Figure 2.5, the peak convergence time transitions from mean degree $\langle k \rangle = 2$ to mean degree $\langle k \rangle = 3$ in adaptive networks. Although it takes far longer for adaptive networks to converge, the trend of convergence time with respect to mean degree is roughly the same as that of the static case.

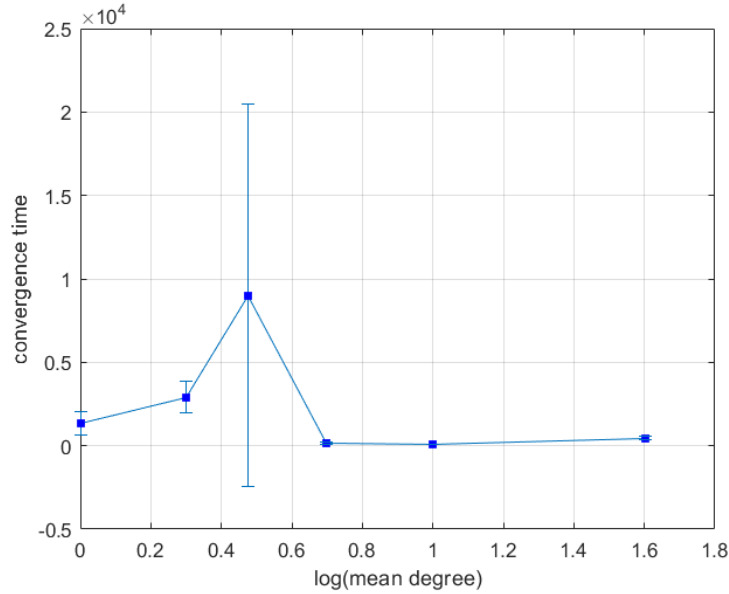


Figure 3.4: Convergence time of adaptive Erdos-Renyi networks with different mean degrees, averaged over 5 replications ($\alpha=0.5$, $d=0.2$, $N=1000$, $p=1$)

We also look at the final states of convergence for networks depicted in Figure 3.4. Compared with its static counterpart, the final state of adaptive networks of mean degree $\langle k \rangle = 1$ is less messy in the sense that there exist fewer clusters at the end. This phenomenon is similar to our observation of fewer floating nodes in the extreme case of maximum node update speed and slow link update speed in Section 3.2. Because link rewiring allows small clusters to find nodes with whom to communicate, these clusters thus might merge together, leading to a decreased total number of

clusters at the final state. At mean degree $\langle k \rangle = 2$, the network converges to a final state of one major cluster that contains approximately 75% of the nodes in the network. We see the transition of final state from one major cluster to two major clusters at mean degree $\langle k \rangle = 3$, where out of 5 replications, two of them converge quickly to two clusters and the rest slowly evolve to one major cluster. This discrepancy in convergence time accounts for the large standard deviation we find at mean degree $\langle k \rangle = 3$ in Figure 3.4. Finally for the rest of the mean degrees in Figure 3.4, the network maintains a final state of two major clusters, though there could exist floating nodes outside the major two.

In Kozma’s paper [10], they find that with rapid speed of link rewiring, a larger value of communication threshold d is necessary to achieve consensus: nodes can more easily search for others with whom they can communicate, and break connections with the ones with too different opinions, so that the formation of different clusters is favored. Although the conclusion is about varying values of communication threshold d , their theory is still applicable to explain the final states of adaptive networks with different mean degrees. For adaptive networks at low degrees, because of the possibility of link rewiring, nodes who would be isolated or remain in small groups on a static network may manage to find nodes with whom to communicate and thus enter a major cluster at the final state. This process of absorbing more nodes into the major cluster continues until at mean degree $\langle k \rangle = 3$, when nodes keep searching for similar nodes and breaking ineffective connections, the large cluster becomes hard to maintain, and can be easily broken. Therefore, adaptation fosters the formation of larger clusters at small mean degrees, while it promotes the division of these clusters at relatively large mean degrees. Depending on which force dominates the opinion evolution, the final states of the networks may look differently. To some extent, the final state of two major clusters at mean degree $\langle k \rangle = 5$ and larger can be seen as an equilibrium balanced by the two forces intrinsic to adaptation.

Chapter 4

Conclusions

In this thesis, we have studied the process of opinion evolution on static and adaptive networks through the investigation of a simple model of opinion dynamics. We have considered the node opinions as a continuous variable on $[0, 1]$, reflecting a possible spectrum of real world opinions. A node's opinion is updated to become more similar to a randomly selected neighbor's opinion, provided that the neighbor's opinion differs by less than a communication threshold d .

When the nodes are connected on a static complete network, we have found that the network reaches a final state of consensus if the communication threshold $d = 0.5$, whereas it converges to two major clusters if we let the communication threshold $d = 0.2$. In the latter case, there could exist some floating nodes outside the two major clusters. We have shown that the convergence time of complete static networks increases linearly as the network size increases, because the convergence time is proportional to the mean degree of the network. Changing the network structure from complete graphs to Erdos-Renyi graphs, we have examined the convergence time of Erdos-Renyi networks with different mean degrees, and the final states of the networks after they are fully synchronized: when the network gets more sparse, the two major clusters that we see in complete graphs gradually lose their nodes due to the decreasing network connectivity, until the network contains only floating clusters at mean degree $\langle k \rangle = 1$. We have also investigated the facilitating effect of smaller values of convergence parameter α on forming consensus at the final state, and that the overall opinion difference in the network decreases by roughly following an exponential decay with rate α .

Next, in contrast to the static networks with fixed structures, we have incorporated the changing nature of the interpersonal relations in real-life social networks. In addition to the opinion dynamics, links that do not communicate due to divergent opinions may be broken with a rewiring probability p and new links are randomly created in the network. In this way, the network changes in an adaptive manner, which combines the topological evolution of the network with dynamics in the network nodes. We have looked at the extreme case of maximum node update speed and slow link update speed, and the other extreme case of maximum link update speed and slow node update speed. In the former case, we have found that by enhancing the communication between nodes with similar opinions, adaptation reduces the number of floating nodes in the network, and promotes the size of major clusters. Our revisit of the effect of convergence parameter α on adaptive

networks also demonstrates that the incorporation of adaptation facilitates achieving consensus in the population. However, a closer look at the final states of adaptive Erdos-Renyi networks with different mean degrees reveals that adaptation fosters the formation of larger clusters at small mean degrees, while it promotes the division of these clusters at comparatively large mean degrees. Depending on how well connected the network is, the dominant effect of adaptation on the model will be different.

Future investigation may consider predicting the model that we have studied in this thesis theoretically. In Demiral's paper [12], they employ moment-closure approximations to analyze adaptive networks with discrete opinion variables. Possibly new approaches can be tailored to predict adaptive networks with continuous opinions. Another direction of future work considers making more realistic modifications of the model. Similar to the work of Malik and Mucha [6], instead of a constant link rewiring probability p , we can assign a distribution of probability for the network to delete non-communicating links, and replace the random link creation with a step that prefers rewiring to nodes that are already close within the network. In this way, the model will incorporate the heterogeneity of influence for individuals in social networks and the phenomenon that friends of friends have a higher likelihood of also being friends.

Chapter 5

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